Quantum and classical areas of black hole thermodynamics

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Abstract

Most calculations of black hole entropy in loop quantum gravity indicate a term proportional to the area eigenvalue A with a correction involving the logarithm of A. This violates the additivity of the entropy. An entropy proportional to A, with a correction term involving the logarithm of the classical area k, which is consistent with the additivity of entropy, is derived in both U(1) and SU(2) formulations.

1 Introduction

A black hole horizon hides what is inside. This lack of information was interpreted early on as a sign of an entropy [1]. The interpretation was strengthened by the fact that the area of a black hole horizon tends to increase, just like an entropy. These ideas led to black hole thermodynamics. Later, the calculation of the temperature at which black holes radiate in quantum theory [2] determined the scale of the entropy S in terms of the area A and led to a precise expression

$$S = \frac{A}{4\hbar G} \tag{1}$$

by integrating the first law of thermodynamics.

While it may have appeared surprising that the entropy is a function of the area instead of the volume, as happens in the case of gases, it must be remembered that areas, like volumes, are also additive. Since state spaces are multiplicative, one may quite generally argue that if the number of states of a black hole is to depend only on the area of the horizon,

$$N(A_1)N(A_2) = N(A_1 + A_2). \tag{2}$$

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This can hold if one considers a composite black hole obtained from two widely separated black holes, so that the horizon has two pieces. The equation implies that

$$N(A) = e^{\lambda A/2}, \tag{3}$$

with some constant λ, implying the area law for the entropy $\log N$. Nothing can be said about λ without a microscopic approach. It should be clear that the temperature of black hole radiation is not an input in this argument. Simply the additive nature of the entropy and the area is used.

Microscopic theories of black holes are supposed to tell us how this entropy arises through a counting of states. This has been achieved to a reasonable degree by loop quantum gravity [3, 4]. Here, the horizon area is described by an operator, with different eigenvalues. The entropy has been found to have a term linear in this area, with an arbitrary proportionality constant, as well as a logarithmic term, which violates the area law [5, 6, 7]. We shall reanalyze the problem carefully and see how the correct answer becomes consistent with the additivity of entropy.

The horizon in the microscopic theory of gravity is supposed to have a set of *punctures* on it. Each puncture is labelled by spin quantum numbers j, m. The area operator has eigenvalues

$$2\sum_p \sqrt{j_p(j_p+1)} = A, \tag{4}$$

where j_p is the angular momentum quantum number associated with the puncture p. This is in a special unit where

$$4\pi\gamma\ell_P^2 = 1,$$

γ being what is called the Immirzi parameter and $\ell_P = \sqrt{\hbar G}$. The total area of the horizon is obtained by adding the areas contributed by the punctures. There is also a holonomy operator whose eigenvalues are of the form $\exp(2\pi i a_p/k)$, where $a_p = 1, 2, ...k$ and the integer k is a measure of the classical area of the horizon (in the unit introduced above) which is also the level of a Chern-Simons theory describing the quantum theory of the horizon [3]. This has to be distinguished from the quantum area A arising from the punctures. The integer a_p, defined modulo k, is required to equal -2 times the angular momentum projection m_p according to the theory. The total spin projection then has to satisfy

$$\sum_p m_p = 0 \bmod k/2 \tag{5}$$

so that the product of the holonomies is unity. We shall investigate the entropy in this theory using Meissner's approach [5] but paying attention to k and find a different, k-dependent logarithmic correction consistent with the additivity of entropy. We shall also generalize that approach to the SU(2) formulation of loop quantum gravity.

2 U(1) loop quantum gravity à la Meissner

Meissner's first recursion relation, where the projection constraint is ignored, may be written as

$$N(A) = \sum_j \sum_m N\Big(A - 2\sqrt{j(j+1)}\Big) + \sum_m \theta(j_{max} - |m|). \tag{6}$$

To understand this, one has to split the set of all distributions of spins into those with only one puncture and those with at least two punctures. If there is only one puncture, its j must satisfy $2\sqrt{j_{max}(j_{max}+1)} = A$, from which j_{max} can be found. When there are at least two punctures, the configurations are again split, depending on the spin j on the first puncture. This puncture can be filled with different values of m for the given j, *i.e.*, in $\sum_m 1$ ways, while the remainder can be filled in $N\Big(A - 2\sqrt{j(j+1)}\Big)$ ways because one puncture has taken away a piece $2\sqrt{j(j+1)}$ from A. The spin j of the first puncture must then be summed over. It may be mentioned that one would expect that $\sum_m 1 = 2j+1$, but the value 2 has been used [5] in this context. Our analysis allows both situations.

It is important to understand that this relation differs from (2) because the area is an operator in loop quantum gravity. The simpler relation can only hold when a unique area can be assigned to a physical system like a black hole.

The current recursion relation is satisfied for large A by an exponential form

$$N(A) \sim \exp(\lambda A/2) \tag{7}$$

with λ satisfying the equation

$$\sum_j \sum_m \exp\Big[-\lambda\sqrt{j(j+1)}\Big] = 1. \tag{8}$$

This fixes λ, which in turn fixes the parameter γ required for reproducing $S = \frac{A}{4\ell_P^2}$.

But one must impose the constraint on angular momentum projection: the number of configurations will be reduced, so that a correction is expected to emerge. One has to introduce the projection M and the reduced number $N(A, M)$ of states. Here M stands for $\sum m$, and $M = 0$ is nominally the case of interest.

Let us take a Fourier transform:

$$N(A, M) = \int_{-2\pi}^{2\pi} e^{-i\omega M} \frac{d\omega}{4\pi} \tilde{N}(A, \omega). \tag{9}$$

For vanishing spin projection,

$$N(A, 0) = \int_{-2\pi}^{2\pi} \frac{d\omega}{4\pi} \tilde{N}(A, \omega). \tag{10}$$

The recursion relation involving M is

$$N(A, M) = \sum_j \sum_m N(A - 2\sqrt{j(j+1)}, M - m) + \theta\big(A - 2\sqrt{|M|(|M|+1)}\big). \quad (11)$$

All values of M have to be admitted in the calculation because the numbers N with fixed M do not close. This relation is understood in a way similar to the earlier one. The distributions of punctures are split into those with only one puncture and those with at least two. If there is one puncture, its j is related to A, and whether it can have a projection M or not depends on whether $A \geq 2\sqrt{|M|(|M|+1)}$. If there are two or more punctures, the spin j of the first one is considered and the remainder has A reduced to $A - 2\sqrt{|M|(|M|+1)}$ and M reduced to $M - m$.

The Fourier transform of the recursion relation involving M is

$$\begin{aligned} \int \frac{d\omega}{4\pi} e^{-i\omega M} \tilde{N}(A, \omega) &= \sum_j \sum_m \int \frac{d\omega}{4\pi} e^{-i\omega(M-m)} \tilde{N}(A - 2\sqrt{j(j+1)}, \omega) \\ &+ \theta\big(A - 2\sqrt{|M|(|M|+1)}\big). \end{aligned} \quad (12)$$

One sees that if A is large, it is satisfied by

$$\tilde{N}(A, \omega) \sim \exp(\lambda(\omega)A/2), \quad (13)$$

where $\lambda(\omega)$ obeys

$$1 = \sum_j \exp\Big[-\lambda(\omega)\sqrt{j(j+1)}\Big] \sum_m e^{i\omega m}. \quad (14)$$

For $\omega = 0$, the equation for $\lambda(\omega)$ reduces to that for λ, so $\lambda(0) = \lambda$. This yields the dominant contribution $\exp(\lambda A/2)$ seen above. For small ω, $\lambda(\omega)$ falls quadratically, like $\lambda - c\omega^2$, say, and the ω integral

$$N(A, 0) \sim \int_{-2\pi}^{2\pi} \frac{d\omega}{4\pi} \exp[\lambda(\omega)A/2] = \int_{-2\pi}^{2\pi} \frac{d\omega}{4\pi} \exp[(\lambda - c\omega^2)A/2] \quad (15)$$

becomes a gaussian, which is seen to be proportional to $A^{-1/2}$ for large A by scaling:

$$N(A, 0) \propto \frac{\exp(\lambda A/2)}{A^{1/2}}. \quad (16)$$

This indicates a correction $-\frac{1}{2}\log A$ [5] in the entropy, which violates the area law.

Now we take into account the possibility of M being equal to zero only *modulo* $k/2$, which is the actual requirement. One may approximate the integral for spin projection M:

$$N(A, M) = N(A) e^{-M^2/(2cA)}. \quad (17)$$

Since $M = 0 \bmod k/2$, one has to sum over the values $M = rk/2$, where $r = 0, \pm 1, \pm 2, \ldots$, and there arises a factor

$$\sum_r e^{-r^2 k^2/(8cA)} \quad (18)$$

which, on approximation by an integral over r, is seen to involve a factor $\sqrt{A}/k$, cancelling the square root in $N(A)$. Thus,

$$N_{corr}(A) = \frac{1}{k}\exp(\frac{\lambda A}{2}), \tag{19}$$

indicating that the log correction in A is absent, but there is a correction involving the *classical area* k.

More accurately, one has to add up $N(A) = \sum_r N(A, rk/2)$. This means

$$N(A) = \int_{-2\pi}^{2\pi} \sum_r e^{-irk\omega/2} \frac{d\omega}{4\pi} \tilde{N}(A,\omega). \tag{20}$$

The sum over r of $e^{-irk\omega/2}$ is a geometric series. For $k\omega = 0 \bmod 4\pi$, it becomes a sum of terms all equal to unity and hence diverges. For other values of $k\omega$, it vanishes because of oscillations. This in fact produces $4\pi \sum_s \delta(k\omega - 4s\pi)$, where s goes over all integers.

Because of the bounded range of ω, s takes a finite number of values:

$$N(A) \sim \frac{1}{k} \sum_{s=-k/2}^{k/2} \exp(\frac{\lambda(4s\pi/k)A}{2}), \tag{21}$$

implying that the entropy is a sum of a finite number of exponentials and thus has no scope for a logarithmic correction in A [8], though there is a correction $\log k$ involving the classical area k. This is exponential in the area, but does the correction violate the requirements?

We have to reformulate the requirement of an additive entropy. If the number of states is allowed to depend on an area and an additional additive variable which we call M, the condition (2) gets modified to

$$\begin{aligned} \int \frac{d\omega}{4\pi} e^{-i\omega M} \tilde{N}(A_1 + A_2, \omega) &= \sum_m \int \frac{d^2\omega}{(4\pi)^2} e^{-i\omega_1 m - i\omega_2 (M-m)} \tilde{N}(A_1,\omega_1)\tilde{N}(A_2,\omega_2) \\ &= \int \frac{d\omega}{4\pi} e^{-i\omega M} \tilde{N}(A_1,\omega)\tilde{N}(A_2,\omega). \end{aligned} \tag{22}$$

This is satisfied by (13), with no constraint on $\lambda(\omega)$. With the help of a condition of M vanishing modulo $k/2$, we may reach (21) again. This yields a correction $\log k$ in the entropy. Thus the k correction obtained in loop quantum gravity can be accommodated in the general formalism through M.

3 SU(2) loop quantum gravity

In the alternative SU(2) formulation of loop quantum gravity [9], the number of states for a distribution of spins over punctures arises from properties of $\mathrm{SU}_q(2)$ as

$$N = \frac{2}{k+2} \sum_{a=1}^{k+1} \sin^2 \frac{a\pi}{k+2} \prod_p \left[\frac{\sin \frac{a\pi(2j_p+1)}{k+2}}{\sin \frac{a\pi}{k+2}} \right], \tag{23}$$

where the product is over punctures p.

Note that N can be written as

$$N = \sum_{a=1}^{k+1} N_a, \tag{24}$$

where each N_a depends on the angular momentum quantum numbers at the different punctures:

$$N_a = \frac{2}{k+2} \sin^2 \frac{a\pi}{k+2} \prod_p \frac{\sin \frac{(2j_p+1)a\pi}{k+2}}{\sin \frac{a\pi}{k+2}}. \tag{25}$$

This may be further split into

$$\begin{aligned} N_a &= \frac{2}{k+2} \sin^2 \frac{a\pi}{k+2} \bar{N}_a, \\ \bar{N}_a &= \prod_p \frac{\sin \frac{(2j_p+1)a\pi}{k+2}}{\sin \frac{a\pi}{k+2}} \equiv \prod_p [2j_p + 1]_a. \end{aligned} \tag{26}$$

With this notation, Meissner's relation looks like

$$\bar{N}_a(A) = \sum_j [2j+1]_a \bar{N}_a(A - 2\sqrt{j(j+1)}) + [\sqrt{A^2+1}]_a. \tag{27}$$

The last term comes from $2j_{max}+1$. For large A, this leads to the solution

$$\bar{N}_a = \exp(\lambda_a A/2), \tag{28}$$

with

$$\sum_{j=\frac{1}{2}}^{k/2} [2j+1]_a e^{-\lambda_a \sqrt{j(j+1)}} = 1. \tag{29}$$

Thus

$$N = \sum_a N_a = \sum_a \frac{2}{k+2} \sin^2 \frac{a\pi}{k+2} \exp(\lambda_a A/2), \tag{30}$$

is essentially a sum of terms each increasing exponentially with A and is dominated for large A by the largest λ_a, which is expected to occur for small $\frac{a}{k+2}$ as this makes $[2j+1]_a$ largest for each j. Thus the dependence on A is purely exponential [10]. Note that there is a $3\log(k+2)$ correction much like the U(1) case, the number 3 taking into account the factor $\sin^2 \frac{a\pi}{k+2}$, which becomes $(\frac{a\pi}{k+2})^2$ for small $\frac{a}{k+2}$. The coefficient 3 is different from the earlier calculations, but the ratio of U(1) and SU(2) remains the same. This is related to the fact that the bigger group has three times as many generators as the smaller group.

4 Conclusion

To summarize, the laws of black hole mechanics suggested $S \propto A$. The black hole radiation temperature indicated $S = \frac{A}{4\ell_P^2}$. Loop quantum gravity has earlier indicated $S = \frac{A}{4\ell_P^2} - \frac{1}{2}\log A$ and $S = \frac{A}{4\ell_P^2} - \frac{3}{2}\log A$ in the U(1) and SU(2) formulations respectively. Now we find $S = \frac{A}{4\ell_P^2} - \log k$ and $S = \frac{A}{4\ell_P^2} - 3\log(k+2)$ in the U(1) and the SU(2) formulations respectively. While the older corrections violate the area law, the newer results are consistent with the multiplicativity of the number of states when allowed to depend on another variable besides the area.

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